くわしい ルジャンドル陪関数

1. ルジャンドル陪微分方程式とルジャンドル陪関数

次式はルジャンドルの陪微分方程式（associated Legendre differential equation）と言われる常微分方程式である．媒介変数の n は次数，m は位数とも言う．物理学の問題，特に楕円体に関する境界値問題に現れる．ルジャンドル（Adrien Marie Legendre）はフランスの数学者（1752-1833）である．本書における変数 x は実数，媒介変数 n, m は整数で，$m \le n$ とする．

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{df_{nm}}{dx}\right)+\left(n\left(n+1\right)-\frac{m^2}{1-x^2}\right)f_{nm}=0 \tag{1-1}$$

左辺の中の微分を進めると，

$$\left(1-x^2\right)\frac{d^2 f_{nm}}{dx^2}-2x\frac{df_{nm}}{dx}+\left(n\left(n+1\right)-\frac{m^2}{1-x^2}\right)f_{nm}=0 \tag{1-2}$$

とも表される．特に $m=0$ の場合はルジャンドル微分方程式に一致する．この微分方程式には独立した二種類の解が存在する．一つは記号 $P_{nm}\left(x\right)$ で表される第 1 種ルジャンドル陪関数（associated Legendre function of the first kind）であり，もう一つは記号 $Q_{nm}\left(x\right)$ で表される第 2 種ルジャンドル陪関数（associated Legendre function of the second kind）である．ルジャンドル陪微分方程式の一般解は，$P_{nm}\left(x\right)$ と $Q_{nm}\left(x\right)$ を用いて次式のように表される．

$$f_{nm}\left(x\right)=c_1 P_{nm}\left(x\right)+c_2 Q_{nm}\left(x\right) \tag{1-3}$$

ここに，c_1, c_2 は任意の定数である．符号を変えた $-P_{nm}\left(x\right), -Q_{nm}\left(x\right)$ もルジャンドル陪微分方程式の解である．符号の定義は文献によって異なるので注意を要する．

以下，$P_{nm}\left(x\right)$ を具体的に求める方法を三通り，$Q_{nm}\left(x\right)$ を具体的に求める方法を二通り記す．

2. 第 1 種ルジャンドル陪関数

2.1 微分して計算する方法

$P_{nm}(x)$ は次式の微分計算から求めることができる．右辺の $P_n(x)$ は第 1 種ルジャンドル関数である．

$$P_{nm}(x)=\left(1-x^2\right)^{m/2}\frac{d^m P_n(x)}{dx^m},\quad |x|\le 1 \tag{1-4}$$

$$P_{nm}(x)=\left(x^2-1\right)^{m/2}\frac{d^m P_n(x)}{dx^m},\quad |x|\ge 1 \tag{1-5}$$

式(1-4) を使って，$|x|<1,\quad n=0,1,2,3,\quad m=0,1,2,3$ の $P_{nm}(x)$ を計算すると，

$$\frac{d^1 P_1(x)}{dx^1}=\frac{d}{dx}x=1$$

$$P_{11}(x)=\left(1-x^2\right)^{1/2}\frac{d^1 P_1(x)}{dx^1}=\left(1-x^2\right)^{1/2} \tag{1-6}$$

$$\frac{d^1 P_2(x)}{dx^1}=\frac{d}{dx}\frac{3x^2-1}{2}=3x$$

$$\frac{d^2 P_2(x)}{dx^2}=\frac{d}{dx}(3x)=3$$

$$P_{21}(x)=\left(1-x^2\right)^{1/2}\frac{d^1 P_2(x)}{dx^1}=3x\left(1-x^2\right)^{1/2} \tag{1-7}$$

$$P_{22}(x)=\left(1-x^2\right)^{2/2}\frac{d^2 P_2(x)}{dx^2}=3\left(1-x^2\right) \tag{1-8}$$

$$\frac{d^1 P_3(x)}{dx^1}=\frac{d}{dx}\frac{5x^3-3x}{2}=\frac{15x^2-3}{2}$$

$$\frac{d^2 P_3(x)}{dx^2}=\frac{d}{dx}\frac{15x^2-3}{2}=15x$$

$$\frac{d^3 P_3(x)}{dx^3}=\frac{d}{dx}15x=15$$

$$P_{31}(x)=\left(1-x^2\right)^{1/2}\frac{d^1 P_3(x)}{dx^1}=\frac{15x^2-3}{2}\left(1-x^2\right)^{1/2} \tag{1-9}$$

$$P_{32}(x)=\left(1-x^2\right)^{2/2}\frac{d^2 P_3(x)}{dx^2}=15x\left(1-x^2\right) \tag{1-10}$$

$$P_{33}(x) = \left(1-x^2\right)^{3/2} \frac{d^3 P_3(x)}{dx^3} = 15\left(1-x^2\right)^{3/2} \tag{1-11}$$

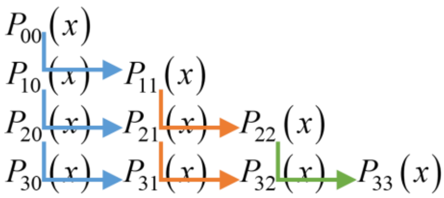
となる．$|x| \geq 1$ の場合も式(1-5) を使って同様に計算でき，結果は$\left(1-x^2\right)^{m/2}$ を$\left(x^2-1\right)^{m/2}$ に置き換えた式となる．

2.2 漸化式から計算する方法

$P_{nm}(x)$ は次式の漸化式を使って計算することもできる．なお，$P_{n,0}(x) = P_n(x)$ である．

$$P_{n,m+1}(x) = \frac{-(n-m)xP_{nm}(x) + (n+m)P_{n-1,m}(x)}{\sqrt{1-x^2}}, \quad |x| \leq 1 \tag{1-12}$$

$$P_{n,m+1}(x) = \frac{(n-m)xP_{nm}(x) - (n+m)P_{n-1,m}(x)}{\sqrt{x^2-1}}, \quad |x| \geq 1 \tag{1-13}$$



式(1-12) を使って，$|x| \leq 1, \quad n = 1, 2, 3, \quad m = 1, 2, 3$ の $P_{nm}(x)$ を計算すると，

$$P_{11}(x) = \frac{-(1-0)xP_{10}(x) + (1+0)P_{00}(x)}{\sqrt{1-x^2}} = \frac{-x \cdot x + 1}{\sqrt{1-x^2}} = \sqrt{1-x^2} \tag{1-14}$$

$$\begin{aligned} P_{21}(x) &= \frac{-(2-0)xP_{20}(x) + (2+0)P_{10}(x)}{\sqrt{1-x^2}} = \frac{-x\left(3x^2-1\right) + 2x}{\sqrt{1-x^2}} \\ &= \frac{x\left(3-3x^2\right)}{\sqrt{1-x^2}} = 3x\sqrt{1-x^2} \end{aligned} \tag{1-15}$$

$$\begin{aligned} P_{22}(x) &= \frac{-(2-1)xP_{21}(x) + (2+1)P_{11}(x)}{\sqrt{1-x^2}} \\ &= \frac{-x \cdot 3x\sqrt{1-x^2} + 3\sqrt{1-x^2}}{\sqrt{1-x^2}} = 3 - 3x^2 = 3\left(1-x^2\right) \end{aligned} \tag{1-16}$$

$$
\begin{aligned}
P_{31}(x) &= \frac{-(3-0)xP_{30}(x)+(3+0)P_{20}(x)}{\sqrt{1-x^2}} = \frac{3}{\sqrt{1-x^2}}\left(-x\cdot\frac{5x^3-3x}{2}+\frac{3x^2-1}{2}\right) \\
&= \frac{3\left(-5x^4+3x^2+3x^2-1\right)}{2\sqrt{1-x^2}} = \frac{3\left(-5x^4+6x^2-1\right)}{2\sqrt{1-x^2}} = \frac{3\left(1-x^2\right)\left(-1+5x^2\right)}{2\sqrt{1-x^2}} \\
&= \frac{3\left(5x^2-1\right)}{2}\sqrt{1-x^2} = \frac{15x^2-3}{2}\sqrt{1-x^2}
\end{aligned}
\tag{1-17}
$$

$$
\begin{aligned}
P_{32}(x) &= \frac{-(3-1)xP_{31}(x)+(3+1)P_{21}(x)}{\sqrt{1-x^2}} \\
&= \frac{1}{\sqrt{1-x^2}}\left(-2x\cdot\frac{15x^3-3}{2}\sqrt{1-x^2}+4\cdot 3x\sqrt{1-x^2}\right) \\
&= -15x^4+3x+12x = 15x\left(1-x^2\right)
\end{aligned}
\tag{1-18}
$$

$$
\begin{aligned}
P_{33}(x) &= \frac{-(3-2)xP_{32}(x)+(3+2)P_{22}(x)}{\sqrt{1-x^2}} \\
&= \frac{1}{\sqrt{1-x^2}}\left(-x\cdot 15x\left(1-x^2\right)+5x\cdot 3\left(1-x^2\right)\right) \\
&= \sqrt{1-x^2}\left(-15x^2+15x\right) = 15\left(1-x^2\right)^{3/2}
\end{aligned}
\tag{1-19}
$$

となる．$|x|\geq 1$ の場合も式(1-13) を使って同様に計算でき，結果は$\left(1-x^2\right)^{m/2}$ を$\left(x^2-1\right)^{m/2}$ に置き換えた式となる．

2.3 直接計算する方法

$P_{nm}(x)$ は次式から直接計算することもできる．

$$
P_{nm}(x) = \frac{1}{2^n}\left(1-x^2\right)^{m/2}\sum_{k=0}^{\mathrm{int}\left[(n-m)/2\right]}(-1)^k\frac{(2n-2k)!}{k!(n-k)!(n-m-2k)!}x^{n-m-2k},\quad |x|\leq 1 \tag{1-20}
$$

$$
P_{nm}(x) = \frac{1}{2^n}\left(x^2-1\right)^{m/2}\sum_{k=0}^{\mathrm{int}\left[(n-m)/2\right]}(-1)^k\frac{(2n-2k)!}{k!(n-k)!(n-m-2k)!}x^{n-m-2k},\quad |x|\geq 1 \tag{1-21}
$$

ここに，$\mathrm{int}\left[(n-m)/2\right]$ は $(n-m)/2$ 以下の最大の整数である．

式(1-20) を使って，$|x|\leq 1,\quad n=1,2,3,\quad m=1,2,3$ の $P_{nm}(x)$ を計算すると，

$$
\begin{aligned}
P_{11}(x) &= \frac{1}{2^1}\left(1-x^2\right)^{1/2}\sum_{k=0}^{0}(-1)^k \frac{(2\cdot 1-2k)!}{k!(1-k)!(1-1-2k)!}x^{1-1-2k} \\
&= \frac{1}{2^1}\left(1-x^2\right)^{1/2}(-1)^0 \frac{(2\cdot 1-2\cdot 0)!}{0!(1-0)!(1-1-2\cdot 0)!}x^{1-1-2\cdot 0} \\
&= \frac{1}{2}\left(1-x^2\right)^{1/2}\cdot 1\cdot \frac{2}{1\cdot 1\cdot 1}x^0 = \left(1-x^2\right)^{1/2}
\end{aligned}
\tag{1-22}
$$

$$
\begin{aligned}
P_{21}(x) &= \frac{1}{2^2}\left(1-x^2\right)^{1/2}\sum_{k=0}^{0}(-1)^k \frac{(2\cdot 2-2k)!}{k!(2-k)!(2-1-2k)!}x^{2-1-2k} \\
&= \frac{1}{2^2}\left(1-x^2\right)^{1/2}(-1)^0 \frac{(2\cdot 2-2\cdot 0)!}{0!(2-0)!(2-1-2\cdot 0)!}x^{2-1-2\cdot 0} \\
&= \frac{1}{4}\left(1-x^2\right)^{1/2}\cdot 1\cdot \frac{24}{1\cdot 2\cdot 1}x^1 = 3x\left(1-x^2\right)^{1/2}
\end{aligned}
\tag{1-23}
$$

$$
\begin{aligned}
P_{22}(x) &= \frac{1}{2^2}\left(1-x^2\right)^{2/2}\sum_{k=0}^{0}(-1)^k \frac{(2\cdot 2-2k)!}{k!(2-k)!(2-2-2k)!}x^{2-2-2k} \\
&= \frac{1}{2^2}\left(1-x^2\right)(-1)^0 \frac{(2\cdot 2-2\cdot 0)!}{0!(2-0)!(2-2-2\cdot 0)!}x^{2-2-2\cdot 0} \\
&= \frac{1}{4}\left(1-x^2\right)\cdot 1\cdot \frac{24}{1\cdot 2\cdot 1}x^0 = 3\left(1-x^2\right)
\end{aligned}
\tag{1-24}
$$

$$
\begin{aligned}
P_{31}(x) &= \frac{1}{2^3}\left(1-x^2\right)^{1/2}\sum_{k=0}^{1}(-1)^k \frac{(2\cdot 3-2k)!}{k!(3-k)!(3-1-2k)!}x^{3-1-2k} \\
&= \frac{1}{8}\left(1-x^2\right)^{1/2}\sum_{k=0}^{1}(-1)^k \frac{(6-2k)!}{k!(3-k)!(2-2k)!}x^{2-2k} \\
&= \frac{1}{8}\left(1-x^2\right)^{1/2}\left(\frac{(6-2\cdot 0)!}{0!(3-0)!(2-2\cdot 0)!}x^{2-2\cdot 0} - \frac{(6-2\cdot 1)!}{1!(3-1)!(2-2\cdot 1)!}x^{2-2\cdot 1}\right) \\
&= \frac{1}{8}\left(1-x^2\right)^{1/2}\left(\frac{720}{1\cdot 6\cdot 2}x^2 - \frac{24}{1\cdot 2\cdot 1}x^0\right) = \frac{1}{8}\left(1-x^2\right)^{1/2}\left(60x^2-12\right) \\
&= \frac{1}{2}\left(1-x^2\right)^{1/2}\left(15x^2-3\right)
\end{aligned}
\tag{1-25}
$$

$$
\begin{aligned}
P_{32}(x) &= \frac{1}{2^3}\left(1-x^2\right)^{2/2}\sum_{k=0}^{0}(-1)^k \frac{(2\cdot 3-2k)!}{k!(3-k)!(3-2-2k)!}x^{3-2-2k} \\
&= \frac{1}{8}\left(1-x^2\right)(-1)^0 \frac{(6-2\cdot 0)!}{0!(3-0)!(1-2\cdot 0)!}x^{1-2\cdot 0} \\
&= \frac{1}{8}\left(1-x^2\right)\cdot 1\cdot \frac{720}{1\cdot 6\cdot 1}x = \frac{1}{8}\left(1-x^2\right)\cdot 120x \\
&= 15\left(1-x^2\right)x
\end{aligned} \tag{1-26}
$$

$$
\begin{aligned}
P_{33}(x) &= \frac{1}{2^3}\left(1-x^2\right)^{3/2}\sum_{k=0}^{0}(-1)^k \frac{(2\cdot 3-2k)!}{k!(3-k)!(3-3-2k)!}x^{3-3-2k} \\
&= \frac{1}{8}\left(1-x^2\right)^{3/2}(-1)^0 \frac{(6-2\cdot 0)!}{0!(3-0)!(0-2\cdot 0)!}x^{0-2\cdot 0} \\
&= \frac{1}{8}\left(1-x^2\right)^{3/2}\cdot 1\cdot \frac{720}{1\cdot 6\cdot 1} = \frac{1}{8}\left(1-x^2\right)^{3/2}\cdot 120 \\
&= 15\left(1-x^2\right)^{3/2}
\end{aligned} \tag{1-27}
$$

となる． $|x|\geq 1$ の場合も式(1-21) を使って同様に計算でき，結果は$\left(1-x^2\right)^{m/2}$ を$\left(x^2-1\right)^{m/2}$に置き換えた式となる．

2.4 第 1 種ルジャンドル陪関数のまとめ

$$P_{11}(x)=\left(1-x^2\right)^{1/2}$$
$$P_{21}(x)=3x\left(1-x^2\right)^{1/2}$$
$$P_{22}(x)=3\left(1-x^2\right)$$
$$P_{31}(x)=\frac{15x^2-3}{2}\left(1-x^2\right)^{1/2}$$
$$P_{32}(x)=15x\left(1-x^2\right)$$
$$P_{33}(x)=15\left(1-x^2\right)^{3/2}$$

表 1　第 1 種ルジャンドル陪関数 $|x|\le 1$

$$P_{11}(x)=\left(x^2-1\right)^{1/2}$$
$$P_{21}(x)=3x\left(x^2-1\right)^{1/2}$$
$$P_{22}(x)=3\left(x^2-1\right)$$
$$P_{31}(x)=\frac{15x^2-3}{2}\left(x^2-1\right)^{1/2}$$
$$P_{32}(x)=15x\left(x^2-1\right)$$
$$P_{33}(x)=15\left(x^2-1\right)^{3/2}$$

表 2　第 1 種ルジャンドル陪関数 $P_{nm}(x)$　$|x|\ge 1$

$|x|\le 1$ における $P_{nm}(x)$ を下図に示す．$n+m$ が偶数の場合は偶関数であり，奇数の場合は奇関数である．偶関数とは $f(-x)=f(x)$ となる関数であり，そのグラフは線対称となる．奇関数とは $f(-x)=-f(x)$ となる関数であり，そのグラフは点対称となる．

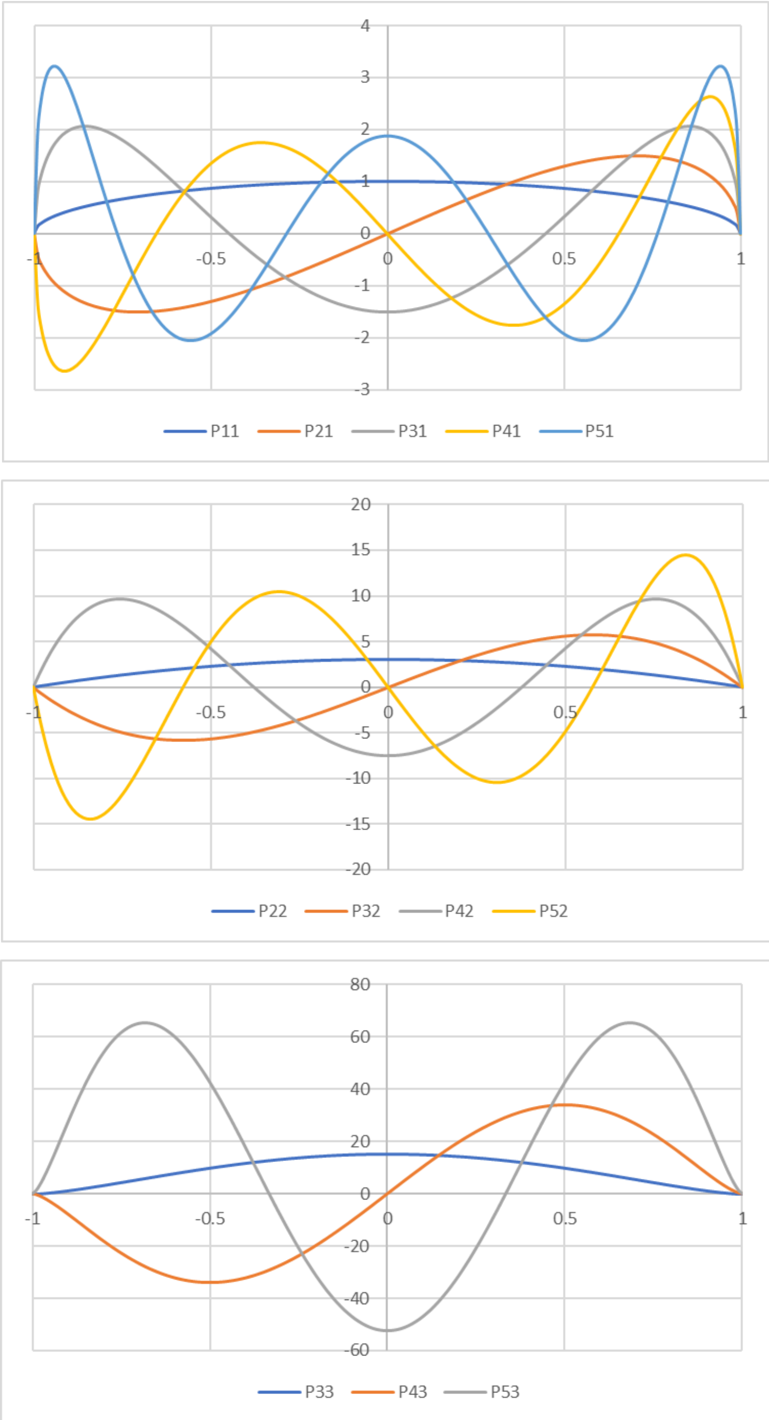


図 1　第 1 種ルジャンドル陪関数 $P_{nm}(x)$　$|x| \le 1$

$P_{nm}(x)$ は $x = \pm 1$ の 2 点を通る．$|x| \le 2$ における $P_{nm}(x)$ を下図に示す．

図 2　第 1 種ルジャンドル陪関数 $P_{nm}(x)$　$|x| \le 2$

$1 \le x \le 10$　の $P_{n,1}(x)$　を対数目盛でプロットすると下図のようになる．$P_{nm}(x)$　の絶対値は単調増加して，$x \to \pm\infty$　では無限大となる．

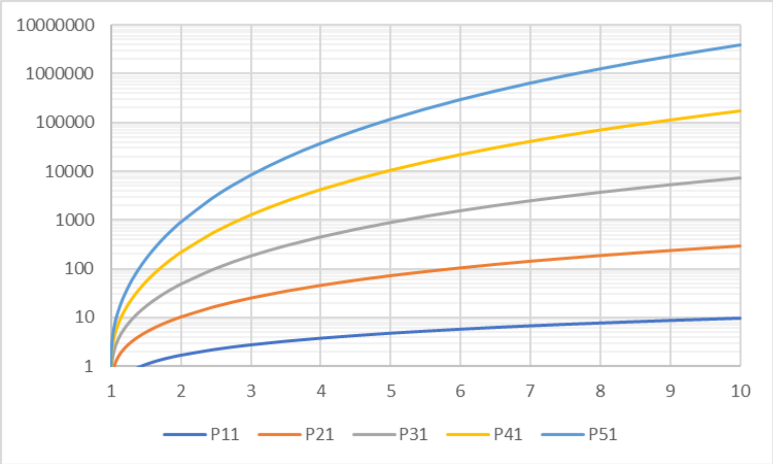


図 3　第 1 種ルジャンドル陪関数 $P_{nm}\left(x\right)$　$1 \le x \le 10$

$x >> 1$　の場合，$P_{nm}\left(x\right)$ は次式のように表され，n 次式に近づく．

$$P_{nm}\left(x\right) = \frac{(2n-1)!!}{\left(n-m\right)!} x^n + O\left(x^{n-2}\right) \tag{1-28}$$

ここに，$\left(2n-1\right)!! = \left(2n-1\right)\left(2n-3\right)\left(2n-5\right)\cdots 3\cdot 1$　であり，$0! = 1$　とする．$O\left(x^{n-2}\right)$　は最高次が $\left(n-2\right)$ 次のなんらかの関数であり，第 1 項と比較すると 2 次小さい．具体的には，

$$\begin{aligned}
P_{11}\left(x\right) &= x + O\left(x^{-1}\right) \\
P_{21}\left(x\right) &= 3x^2 + O\left(x^0\right) \\
P_{22}\left(x\right) &= 3x^2 + O\left(x^0\right) \\
P_{31}\left(x\right) &= \frac{15x^3}{2} + O\left(x^1\right) \\
P_{32}\left(x\right) &= 15x^3 + O\left(x^1\right) \\
P_{33}\left(x\right) &= 15x^3 + O\left(x^1\right)
\end{aligned} \tag{1-29}$$

となる．

2.5 第 1 種ルジャンドル陪関数の証明

$\left|x\right| < 1$,　$n = 1, 2, 3$,　$m = 1, 2, 3$　の $P_{nm}\left(x\right)$　をルジャンドル陪微分方程式(1-2) に実際に代入する．微分計算の途中は省略する．

$P_{11}(x) = \left(1-x^2\right)^{1/2}$ の証明

$$\frac{d}{dx}P_{11}(x) = \frac{-x}{\left(1-x^2\right)^{1/2}}$$

$$\frac{d^2}{dx^2}P_{11}(x) = \frac{-1}{\left(1-x^2\right)^{3/2}}$$

$$\begin{aligned}
&\left(1-x^2\right)\frac{d^2P_{11}(x)}{dx^2} - 2x\frac{dP_{11}(x)}{dx} + \left(1\cdot 2 - \frac{1^2}{1-x^2}\right)P_{11}(x) \\
&= \left(1-x^2\right)\frac{-1}{\left(1-x^2\right)^{3/2}} - 2x\frac{-x}{\left(1-x^2\right)^{1/2}} + \left(2 - \frac{1}{1-x^2}\right)\left(1-x^2\right)^{1/2} \\
&= \frac{-1}{\left(1-x^2\right)^{1/2}} + \frac{2x^2}{\left(1-x^2\right)^{1/2}} + 2\left(1-x^2\right)^{1/2} - \frac{1}{\left(1-x^2\right)^{1/2}} \\
&= \frac{-1+2x^2-1}{\left(1-x^2\right)^{1/2}} + 2\left(1-x^2\right)^{1/2} = -2\left(1-x^2\right)^{1/2} + 2\left(1-x^2\right)^{1/2} = 0
\end{aligned} \tag{A1}$$

$P_{21}(x) = 3x\left(1-x^2\right)^{1/2}$ の証明

$$\frac{d}{dx}P_{21}(x) = \frac{3-6x^2}{\left(1-x^2\right)^{1/2}}$$

$$\frac{d^2}{dx^2}P_{21}(x) = \frac{-9x+6x^3}{\left(1-x^2\right)^{3/2}}$$

$$\begin{aligned}
&\left(1-x^2\right)\frac{d^2P_{21}(x)}{dx^2} - 2x\frac{dP_{21}(x)}{dx} + \left(2\cdot 3 - \frac{1^2}{1-x^2}\right)P_{21}(x) \\
&= \left(1-x^2\right)\frac{-9x+6x^3}{\left(1-x^2\right)^{3/2}} - 2x\frac{3-6x^2}{\left(1-x^2\right)^{1/2}} + \left(6 - \frac{1}{1-x^2}\right)\cdot 3x\left(1-x^2\right)^{1/2} \\
&= \frac{-9x+6x^3}{\left(1-x^2\right)^{1/2}} + \frac{-6x+12x^3}{\left(1-x^2\right)^{1/2}} + 18x\left(1-x^2\right)^{1/2} - \frac{3x}{\left(1-x^2\right)^{1/2}} \\
&= \frac{-18x+18x^3}{\left(1-x^2\right)^{1/2}} + 18x\left(1-x^2\right)^{1/2} = -18x\left(1-x^2\right)^{1/2} + 18x\left(1-x^2\right)^{1/2} = 0
\end{aligned} \tag{A2}$$

$P_{22}(x) = 3(1-x^2)$ の証明

$$\frac{d}{dx}P_{22}(x) = -6x$$

$$\frac{d^2}{dx^2}P_{22}(x) = -6$$

$$\begin{aligned}
&(1-x^2)\frac{d^2P_{22}(x)}{dx^2} - 2x\frac{dP_{22}(x)}{dx} + \left(2\cdot 3 - \frac{2^2}{1-x^2}\right)P_{22}(x) \\
&\quad = (1-x^2)(-6) - 2x(-6x) + \left(6 - \frac{4}{1-x^2}\right)\cdot 3(1-x^2) \\
&\quad = -6 + 6x^2 + 12x^2 + 18 - 18x^2 - 12 = 0
\end{aligned} \tag{A3}$$

$P_{31}(x) = \dfrac{15x^2-3}{2}(1-x^2)^{1/2}$ の証明

$$\frac{d}{dx}P_{31}(x) = \frac{33x - 45x^3}{2(1-x^2)^{1/2}}$$

$$\frac{d^2}{dx^2}P_{31}(x) = \frac{33 - 135x^2 + 90x^4}{2(1-x^2)^{3/2}}$$

$$\begin{aligned}
&(1-x^2)\frac{d^2P_{31}(x)}{dx^2} - 2x\frac{dP_{31}(x)}{dx} + \left(3\cdot 4 - \frac{1^2}{1-x^2}\right)P_{31}(x) \\
&= (1-x^2)\frac{33-135x^2+90x^4}{2(1-x^2)^{3/2}} - 2x\cdot\frac{33x-45x^3}{2(1-x^2)^{1/2}} + \left(12 - \frac{1}{1-x^2}\right)\cdot\frac{15x^2-3}{2}(1-x^2)^{1/2} \\
&= \frac{33-135x^2+90x^4}{2(1-x^2)^{1/2}} + \frac{-33x^2+45x^4}{(1-x^2)^{1/2}} + 6(15x^2-3)(1-x^2)^{1/2} - \frac{15x^2-3}{2(1-x^2)^{1/2}} \\
&= \frac{33-135x^2+90x^4-66x^2+90x^4-15x^2+3}{2(1-x^2)^{1/2}} + 6(15x^2-3)(1-x^2)^{1/2} \\
&= \frac{36-216x^2+180x^4}{2(1-x^2)^{1/2}} + 6(15x^2-3)(1-x^2)^{1/2} \\
&= \frac{12(3-18x^2+15x^4)}{2(1-x^2)^{1/2}} + 6(15x^2-3)(1-x^2)^{1/2} \\
&= 6\frac{(1-x^2)(3-15x^2)}{(1-x^2)^{1/2}} + 6(15x^2-3)(1-x^2)^{1/2} = 0
\end{aligned} \tag{A4}$$

$P_{32}(x)=15x\left(1-x^2\right)$ の証明

$$\frac{d}{dx}P_{32}(x)=15\left(1-3x^2\right)$$

$$\frac{d^2}{dx^2}P_{32}(x)=-90x$$

$$\begin{aligned}&\left(1-x^2\right)\frac{d^2P_{32}(x)}{dx^2}-2x\frac{dP_{32}(x)}{dx}+\left(3\cdot 4-\frac{2^2}{1-x^2}\right)P_{32}(x)\\&=\left(1-x^2\right)\left(-90x\right)-2x\cdot 15\left(1-3x^2\right)+\left(12-\frac{4}{1-x^2}\right)\cdot 15x\left(1-x^2\right)\\&=-90x+90x^3-30x+90x^3+180x\left(1-x^2\right)-60x=0\end{aligned}\tag{A5}$$

$P_{33}(x)=15\left(1-x^2\right)^{3/2}$ の証明

$$\frac{d}{dx}P_{33}(x)=-45x\left(1-x^2\right)^{1/2}$$

$$\frac{d^2}{dx^2}P_{33}(x)=\frac{-45+90x^2}{\left(1-x^2\right)^{1/2}}$$

$$\begin{aligned}&\left(1-x^2\right)\frac{d^2P_{33}(x)}{dx^2}-2x\frac{dP_{33}(x)}{dx}+\left(3\cdot 4-\frac{3^2}{1-x^2}\right)P_{33}(x)\\&=\left(1-x^2\right)\cdot\frac{-45+90x^2}{\left(1-x^2\right)^{1/2}}-2x\cdot\left(-45x\left(1-x^2\right)^{1/2}\right)+\left(12-\frac{9}{1-x^2}\right)\cdot 15\left(1-x^2\right)^{3/2}\\&=\left(1-x^2\right)^{1/2}\left(-45+90x^2+90x^2-135\right)+180\left(1-x^2\right)^{3/2}\\&=\left(1-x^2\right)^{1/2}\left(-180+180x^2\right)+180\left(1-x^2\right)^{3/2}\\&=-180\left(1-x^2\right)^{3/2}+180\left(1-x^2\right)^{3/2}=0\end{aligned}\tag{A6}$$

確かにルジャンドル陪微分方程式を満足している．$|x|\geq 1$ の場合も同様に証明できる．

3. 第 2 種ルジャンドル陪関数

3.1 微分して計算する方法

$Q_{nm}(x)$ は次式の微分計算から求めることができる．右辺の $Q_n(x)$ は第 2 種ルジャンドル関

数である． $P_{nm}\left(x\right)$ の計算式(1-4), (1-5) と同形である．

$$Q_{nm}\left(x\right)=\left(1-x^2\right)^{m/2}\frac{d^mQ_n\left(x\right)}{dx^m},\quad \left|x\right|<1 \tag{2-1}$$

$$Q_{nm}\left(x\right)=\left(x^2-1\right)^{m/2}\frac{d^mQ_n\left(x\right)}{dx^m},\quad \left|x\right|>1 \tag{2-2}$$

$\left|x\right|<1$ の場合の計算の準備として，対数関数 $\ln\left(\frac{1+x}{1-x}\right)$ を微分すると，

$$\begin{aligned}\frac{d}{dx}\left(\ln\left(\frac{1+x}{1-x}\right)\right)&=\frac{1-x}{1+x}\cdot\frac{1\cdot\left(1-x\right)-\left(1+x\right)\cdot\left(-1\right)}{\left(1-x\right)^2}\\&=\frac{1-x}{1+x}\cdot\frac{2}{\left(1-x\right)^2}=\frac{2}{\left(1+x\right)\left(1-x\right)}=\frac{2}{1-x^2}\end{aligned} \tag{2-3}$$

となる．この結果を用いながら， $\left|x\right|<1,\quad n=1,2,3,\quad m=1,2,3$ の $Q_n\left(x\right)$ を計算すると，

$$\begin{aligned}\frac{d^1Q_1\left(x\right)}{dx^1}&=\frac{d}{dx}\left(\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)-1\right)\\&=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{x}{2}\cdot\frac{2}{1-x^2}=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{x}{1-x^2}\\Q_{11}\left(x\right)&=\left(1-x^2\right)^{1/2}\frac{d^1Q_1\left(x\right)}{dx^1}=\left(1-x^2\right)^{1/2}\left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{x}{1-x^2}\right)\\&=\frac{1}{2}\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{x}{\left(1-x^2\right)^{1/2}}\end{aligned} \tag{2-4}$$

$$\begin{aligned}\frac{d^1Q_2\left(x\right)}{dx^1}&=\frac{d}{dx}\left(\frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right)-\frac{3x}{2}\right)\\&=\frac{6x}{4}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-1}{4}\cdot\frac{2}{1-x^2}-\frac{3}{2}\\&=\frac{3x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-1-3\left(1-x^2\right)}{2\left(1-x^2\right)}\\&=\frac{3x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{6x^2-4}{2\left(1-x^2\right)}\\&=\frac{3x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-2}{1-x^2}\end{aligned}$$

$$\frac{d^2Q_2(x)}{dx^2}=\frac{d}{dx}\left(\frac{3x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-2}{1-x^2}\right)$$

$$=\frac{3}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x}{2}\cdot\frac{2}{1-x^2}+\frac{6x\cdot\left(1-x^2\right)-\left(3x^2-2\right)\cdot(-2x)}{\left(1-x^2\right)^2}$$

$$=\frac{3}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x}{1-x^2}+\frac{6x-6x^3+6x^3-4x}{\left(1-x^2\right)^2}$$

$$=\frac{3}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x\left(1-x^2\right)+2x}{\left(1-x^2\right)^2}$$

$$=\frac{3}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{-3x^3+5x}{\left(1-x^2\right)^2}$$

$$Q_{21}(x)=\left(1-x^2\right)^{1/2}\frac{d^1Q_2(x)}{dx^1}=\left(1-x^2\right)^{1/2}\left(\frac{3x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-2}{1-x^2}\right)$$

$$=\frac{3}{2}x\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-2}{\left(1-x^2\right)^{1/2}} \tag{2-5}$$

$$Q_{22}(x)=\left(1-x^2\right)^{2/2}\frac{d^2Q_2(x)}{dx^2}=\left(1-x^2\right)\left(\frac{3}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{-3x^3+5x}{\left(1-x^2\right)^2}\right)$$

$$=\frac{3}{2}\left(1-x^2\right)\ln\left(\frac{1+x}{1-x}\right)+\frac{-3x^3+5x}{1-x^2} \tag{2-6}$$

$$\frac{d^1Q_3(x)}{dx^1}=\frac{d}{dx}\left(\frac{5x^3-3x}{4}\ln\left(\frac{1+x}{1-x}\right)-\frac{15x^2-4}{6}\right)$$

$$=\frac{15x^2-3}{4}\ln\left(\frac{1+x}{1-x}\right)+\frac{5x^3-3x}{4}\cdot\frac{2}{1-x^2}-5x$$

$$=\frac{15x^2-3}{4}\ln\left(\frac{1+x}{1-x}\right)+\frac{5x^3-3x-10x\left(1-x^2\right)}{2\left(1-x^2\right)}$$

$$=\frac{15x^2-3}{4}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^3-13x}{2\left(1-x^2\right)}$$

$$\frac{d^2Q_3(x)}{dx^2}=\frac{d}{dx}\left(\frac{15x^2-3}{4}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^3-13x}{2\left(1-x^2\right)}\right)$$

$$=\frac{30x}{4}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^2-3}{4}\cdot\frac{2}{1-x^2}+\frac{\left(45x^2-13\right)\cdot\left(1-x^2\right)-\left(15x^3-13x\right)\cdot\left(-2x\right)}{2\left(1-x^2\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^2-3}{2\left(1-x^2\right)}+\frac{45x^2-13-45x^4+13x^2+30x^4-26x^2}{2\left(1-x^2\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^2-3}{2\left(1-x^2\right)}+\frac{-15x^4+32x^2-13}{2\left(1-x^2\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{\left(15x^2-3\right)\left(1-x^2\right)-15x^4+32x^2-13}{2\left(1-x^2\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{-15x^4+18x^2-3-15x^4+32x^2-13}{2\left(1-x^2\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{-30x^4+50x^2-16}{2\left(1-x^2\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{-15x^4+25x^2-8}{\left(1-x^2\right)^2}$$

$$\frac{d^3Q_3(x)}{dx^3}=\frac{d}{dx}\left(\frac{15x}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{-15x^4+25x^2-8}{\left(1-x^2\right)^2}\right)$$

$$=\frac{15}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x}{2}\cdot\frac{2}{1-x^2}+\frac{\left(\begin{array}{l}\left(-60x^3+50x\right)\left(1-x^2\right)^2\\ \qquad-\left(-15x^4+25x^2-8\right)\cdot2\left(1-x^2\right)\left(-2x\right)\end{array}\right)}{\left(1-x^2\right)^4}$$

$$=\frac{15}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x}{1-x^2}+\frac{\left(-60x^3+50x\right)\left(1-x^2\right)-\left(-15x^4+25x^2-8\right)\cdot\left(-4x\right)}{\left(1-x^2\right)^3}$$

$$=\frac{15}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x}{1-x^2}+\frac{60x^5-110x^3+50x-60x^5+100x^3-32x}{\left(1-x^2\right)^3}$$

$$=\frac{15}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x}{1-x^2}+\frac{-10x^3+18x}{\left(1-x^2\right)^3}=\frac{15}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x\left(1-x^2\right)^2-10x^3+18x}{\left(1-x^2\right)^3}$$

$$=\frac{15}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x-30x^3+15x^5-10x^3+18x}{\left(1-x^2\right)^3}=\frac{15}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^5-40x^3+33x}{\left(1-x^2\right)^3}$$

$$
\begin{aligned}
Q_{31}(x) &= \left(1-x^2\right)^{1/2} \frac{d^1 Q_3(x)}{dx^1} \\
&= \left(1-x^2\right)^{1/2} \left(\frac{15x^2-3}{4} \ln\left(\frac{1+x}{1-x}\right) + \frac{15x^3-13x}{2\left(1-x^2\right)} \right) \\
&= \frac{15x^2-3}{4} \left(1-x^2\right)^{1/2} \ln\left(\frac{1+x}{1-x}\right) + \frac{15x^3-13x}{2\left(1-x^2\right)^{1/2}}
\end{aligned}
$$

(2-7)

$$
\begin{aligned}
Q_{32}(x) &= \left(1-x^2\right)^{2/2} \frac{d^2 Q_3(x)}{dx^2} \\
&= \left(1-x^2\right) \left(\frac{15x}{2} \ln\left(\frac{1+x}{1-x}\right) + \frac{-15x^4+25x^2-8}{\left(1-x^2\right)^2} \right) \\
&\frac{15}{2} x \left(1-x^2\right) \ln\left(\frac{1+x}{1-x}\right) + \frac{-15x^4+25x^2-8}{1-x^2}
\end{aligned}
$$

(2-8)

$$
\begin{aligned}
Q_{33}(x) &= \left(1-x^2\right)^{3/2} \frac{d^3 Q_3(x)}{dx^3} \\
&= \left(1-x^2\right)^{3/2} \left(\frac{15}{2} \ln\left(\frac{1+x}{1-x}\right) + \frac{15x^5-40x^3+33x}{\left(1-x^2\right)^3} \right) \\
&= \frac{15}{2} \left(1-x^2\right)^{3/2} \ln\left(\frac{1+x}{1-x}\right) + \frac{15x^5-40x^3+33x}{\left(1-x^2\right)^{3/2}}
\end{aligned}
$$

(2-9)

となる．$|x|>1$ の場合の計算準備として対数関数 $\ln\left(\frac{x+1}{x-1}\right)$ を微分すると，

$$
\begin{aligned}
\frac{d}{dx}\left(\ln\left(\frac{x+1}{x-1}\right) \right) &= \frac{x-1}{x+1} \cdot \frac{1\cdot(x-1)-(x+1)\cdot 1}{(x-1)^2} \\
&= \frac{x-1}{x+1} \cdot \frac{-2}{(x-1)^2} = \frac{-2}{(x+1)(x-1)} = \frac{-2}{x^2-1}
\end{aligned}
$$

(2-10)

となる．この結果を用いながら，$|x|>1,\quad n=1,2,3$ の $Q_{nm}(x)$ を計算すると，

$$
\begin{aligned}
\frac{d^1 Q_1(x)}{dx^1} &= \frac{d}{dx}\left(\frac{x}{2} \ln\left(\frac{x+1}{x-1}\right) - 1 \right) \\
&= \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right) + \frac{x}{2} \cdot \frac{-2}{x^2-1} = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) - \frac{x}{x^2-1}
\end{aligned}
$$

$$Q_{11}(x)=\left(x^2-1\right)^{1/2}\frac{d^1Q_1(x)}{dx^1}=\left(x^2-1\right)^{1/2}\left(\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{x}{x^2-1}\right)$$

$$=\frac{1}{2}\left(x^2-1\right)^{1/2}\ln\left(\frac{x+1}{x-1}\right)-\frac{x}{\left(x^2-1\right)^{1/2}} \tag{2-11}$$

$$\frac{d^1Q_2(x)}{dx^1}=\frac{d}{dx}\left(\frac{3x^2-1}{4}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x}{2}\right)$$

$$=\frac{6x}{4}\ln\left(\frac{x+1}{x-1}\right)+\frac{3x^2-1}{4}\cdot\frac{-2}{x^2-1}-\frac{3}{2}$$

$$=\frac{3x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^2-1+3\left(x^2-1\right)}{2\left(x^2-1\right)}$$

$$=\frac{3x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{6x^2-4}{2\left(x^2-1\right)}$$

$$=\frac{3x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^2-2}{x^2-1}$$

$$\frac{d^2Q_2(x)}{dx^2}=\frac{d}{dx}\left(\frac{3x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^2-2}{x^2-1}\right)$$

$$=\frac{3}{2}\ln\left(\frac{x+1}{x-1}\right)+\frac{3x}{2}\cdot\frac{-2}{x^2-1}-\frac{6x\cdot\left(x^2-1\right)-\left(3x^2-2\right)\cdot 2x}{\left(x^2-1\right)^2}$$

$$=\frac{3}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x}{x^2-1}-\frac{6x^3-6x-6x^3+4x}{\left(x^2-1\right)^2}$$

$$=\frac{3}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x\left(x^2-1\right)-2x}{\left(x^2-1\right)^2}$$

$$=\frac{3}{2}\ln\left(\frac{x+1}{x-1}\right)+\frac{3x^3-5x}{\left(x^2-1\right)^2}$$

$$Q_{21}(x)=\left(x^2-1\right)^{1/2}\frac{d^1Q_2(x)}{dx^1}=\left(x^2-1\right)^{1/2}\left(\frac{3x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^2-2}{x^2-1}\right)$$

$$=\frac{3x}{2}\left(x^2-1\right)^{1/2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^2-2}{\left(x^2-1\right)^{1/2}} \tag{2-12}$$

$$Q_{22}(x)=\left(x^2-1\right)^{2/2}\frac{d^2Q_2(x)}{dx^2}=\left(x^2-1\right)\left(\frac{3}{2}\ln\left(\frac{x+1}{x-1}\right)+\frac{3x^3-5x}{\left(x^2-1\right)^2}\right)$$

$$=\frac{3}{2}\left(x^2-1\right)\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^3-5x}{x^2-1} \tag{2-13}$$

$$\frac{d^1Q_3(x)}{dx^1}=\frac{d}{dx}\left(\frac{5x^3-3x}{4}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^2-4}{6}\right)$$

$$=\frac{15x^2-3}{4}\ln\left(\frac{x+1}{x-1}\right)+\frac{5x^3-3x}{4}\cdot\frac{-2}{x^2-1}-5x$$

$$=\frac{15x^2-3}{4}\ln\left(\frac{x+1}{x-1}\right)-\frac{5x^3-3x+10x\left(x^2-1\right)}{2\left(x^2-1\right)}$$

$$=\frac{15x^2-3}{4}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^3-13x}{2\left(x^2-1\right)}$$

$$\frac{d^2Q_3(x)}{dx^2}=\frac{d}{dx}\left(\frac{15x^2-3}{4}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^3-13x}{2\left(x^2-1\right)}\right)$$

$$=\frac{30x}{4}\ln\left(\frac{x+1}{x-1}\right)+\frac{15x^2-3}{4}\cdot\frac{-2}{x^2-1}-\frac{\left(45x^2-13\right)\cdot\left(x^2-1\right)-\left(15x^3-13x\right)\cdot 2x}{2\left(x^2-1\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^2-3}{2\left(x^2-1\right)}-\frac{45x^4-13x^2-45x^2+13-30x^4+26x^2}{2\left(x^2-1\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^2-3}{2\left(x^2-1\right)}-\frac{15x^4-32x^2+13}{2\left(x^2-1\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{\left(15x^2-3\right)\left(x^2-1\right)+15x^4-32x^2+13}{2\left(x^2-1\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^4-18x^2+3+15x^4-32x^2+13}{2\left(x^2-1\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{30x^4-50x^2+16}{2\left(x^2-1\right)^2}$$

$$=\frac{15x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^4-25x^2+8}{\left(x^2-1\right)^2}$$

$$\frac{d^3Q_3(x)}{dx^3}=\frac{d}{dx}\left(\frac{15x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^4-25x^2+8}{\left(x^2-1\right)^2}\right)$$

$$=\frac{15}{2}\ln\left(\frac{x+1}{x-1}\right)+\frac{15x}{2}\cdot\frac{-2}{x^2-1}-\frac{\left(\begin{array}{l}\left(60x^3-50x\right)\left(x^2-1\right)^2\\ \quad-\left(15x^4-25x^2+8\right)\cdot 2\left(x^2-1\right)\cdot 2x\end{array}\right)}{\left(x^2-1\right)^4}$$

$$=\frac{15}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x}{x^2-1}-\frac{\left(60x^3-50x\right)\left(x^2-1\right)-\left(15x^4-25x^2+8\right)\cdot 4x}{\left(x^2-1\right)^3}$$

$$=\frac{15}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x}{x^2-1}-\frac{60x^5-110x^3+50x-60x^5+100x^3-32x}{\left(x^2-1\right)^3}$$

$$=\frac{15}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x}{x^2-1}-\frac{-10x^3+18x}{\left(x^2-1\right)^3}=\frac{15}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x\left(x^2-1\right)^2-10x^3+18x}{\left(x^2-1\right)^3}$$

$$=\frac{15}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^5-30x^3+15x-10x^3+18x}{\left(x^2-1\right)^3}=\frac{15}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^5-40x^3+33x}{\left(x^2-1\right)^3}$$

$$Q_{31}(x)=\left(x^2-1\right)^{1/2}\frac{d^1Q_3(x)}{dx^1}$$

$$=\left(x^2-1\right)^{1/2}\left(\frac{15x^2-3}{4}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^3-13x}{2\left(x^2-1\right)}\right)$$

$$=\frac{15x^2-3}{4}\left(x^2-1\right)^{1/2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^3-13x}{2\left(x^2-1\right)^{1/2}}$$

(2-14)

$$Q_{32}(x)=\left(x^2-1\right)^{2/2}\frac{d^2Q_3(x)}{dx^2}$$

$$=\left(x^2-1\right)\left(\frac{15x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^4-25x^2+8}{\left(x^2-1\right)^2}\right)$$

$$\frac{15}{2}x\left(x^2-1\right)\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^4-25x^2+8}{x^2-1}$$

(2-15)

$$
\begin{aligned}
Q_{33}(x) &= \left(x^2-1\right)^{3/2}\frac{d^3Q_3(x)}{dx^3} \\
&= \left(x^2-1\right)^{3/2}\left(\frac{15}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^5-40x^3+33x}{\left(x^2-1\right)^3}\right) \\
&= \frac{15}{2}\left(x^2-1\right)^{3/2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^5-40x^3+33x}{\left(x^2-1\right)^{3/2}}
\end{aligned}
\tag{2-16}
$$

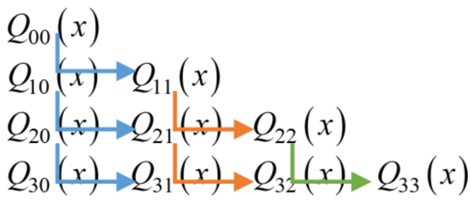
となる．

3.2 漸化式から計算する方法

$Q_{nm}(x)$ は次式の漸化式を使って計算することもできる．なお，$Q_{n,0}(x)=Q_n(x)$ である．$P_{nm}(x)$ の漸化式(1-12), (1-13) と同形である．

$$
Q_{n,m+1}(x)=\frac{-(n-m)xQ_{nm}(x)+(n+m)Q_{n-1,m}(x)}{\sqrt{1-x^2}},\quad |x|<1 \tag{2-17}
$$

$$
Q_{n,m+1}(x)=\frac{(n-m)xQ_{nm}(x)-(n-m)Q_{n-1,m}(x)}{\sqrt{x^2-1}},\quad |x|>1 \tag{2-18}
$$



式(2-17) を使って，$|x|<1,\quad n=1,2,3,\quad m=1,2,3$ の $Q_{nm}(x)$ を計算すると，

$$
\begin{aligned}
Q_{11}(x) &= \frac{-(1-0)xQ_{10}(x)+(1+0)Q_{00}(x)}{\sqrt{1-x^2}} \\
&= \frac{1}{\sqrt{1-x^2}}\left(-x\left(\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)-1\right)+\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right) \\
&= \frac{1}{\sqrt{1-x^2}}\left(\frac{-x^2+1}{2}\ln\left(\frac{1+x}{1-x}\right)+x\right) \\
&= \frac{1}{2}\sqrt{1-x^2}\ln\left(\frac{1+x}{1-x}\right)+\frac{x}{\sqrt{1-x^2}}
\end{aligned}
\tag{2-19}
$$

$$
\begin{aligned}
Q_{21}(x) &= \frac{-(2-0)xQ_{20}(x)+(2+0)Q_{10}(x)}{\sqrt{1-x^2}} \\
&= \frac{1}{\sqrt{1-x^2}}\left(-2x\left(\frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right)-\frac{3}{2}x\right)+2\left(\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)-1\right)\right) \\
&= \frac{1}{\sqrt{1-x^2}}\left(\frac{-x\left(3x^2-1\right)+2x}{2}\ln\left(\frac{1+x}{1-x}\right)+3x^2-2\right) \\
&= \frac{1}{\sqrt{1-x^2}}\left(\frac{-3x^2+3x}{2}\ln\left(\frac{1+x}{1-x}\right)+3x^2-2\right) \\
&= \frac{1}{\sqrt{1-x^2}}\left(\frac{3x\left(1-x^2\right)}{2}\ln\left(\frac{1+x}{1-x}\right)+3x^2-2\right) \\
&= \frac{3}{2}x\sqrt{1-x^2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-2}{\sqrt{1-x^2}} \qquad (2\text{-}20)
\end{aligned}
$$

$$
\begin{aligned}
Q_{22}(x) &= \frac{-(2-1)xQ_{21}(x)+(2+1)Q_{11}(x)}{\sqrt{1-x^2}} \\
&= \frac{1}{\sqrt{1-x^2}}\left(\begin{aligned}&-x\left(\frac{3}{2}x\sqrt{1-x^2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-2}{\sqrt{1-x^2}}\right)\\ &\quad+3\left(\frac{1}{2}\sqrt{1-x^2}\ln\left(\frac{1+x}{1-x}\right)+\frac{x}{\sqrt{1-x^2}}\right)\end{aligned}\right) \\
&= -\frac{3}{2}x^2\ln\left(\frac{1+x}{1-x}\right)-\frac{3x^3-2x}{1-x^2}+\frac{3}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x}{1-x^2} \\
&= \frac{3\left(1-x^2\right)}{2}\ln\left(\frac{1+x}{1-x}\right)-\frac{3x^3-2x-3x}{1-x^2} \\
&= \frac{3\left(1-x^2\right)}{2}\ln\left(\frac{1+x}{1-x}\right)-\frac{3x^3-5x}{1-x^2} \qquad (2\text{-}21)
\end{aligned}
$$

$$Q_{31}(x) = \frac{-(3-0)xQ_{30}(x)+(3+0)Q_{20}(x)}{\sqrt{1-x^2}}$$
$$= \frac{1}{\sqrt{1-x^2}}\left(-3x\left(\frac{5x^3-3x}{4}\ln\left(\frac{1+x}{1-x}\right)-\frac{15x^2-4}{6}\right)+3\left(\frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right)-\frac{3}{2}x\right)\right)$$
$$= \frac{1}{\sqrt{1-x^2}}\left(\frac{-15x^4+9x^2+9x^2-3}{4}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^3-4x}{2}-\frac{9}{2}x\right)$$
$$= \frac{1}{\sqrt{1-x^2}}\left(\frac{-15x^4+18x^2-3}{4}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^3-13x}{2}\right)$$
$$= \frac{1}{\sqrt{1-x^2}}\left(\frac{(1-x^2)(-3+15x^2)}{4}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^3-13x}{2}\right)$$
$$= \frac{15x^2-3}{4}\sqrt{1-x^2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^3-13x}{2\sqrt{1-x^2}} \tag{2-22}$$

$$Q_{32}(x) = \frac{-(3-1)xQ_{31}(x)+(3+1)Q_{21}(x)}{\sqrt{1-x^2}}$$
$$= \frac{1}{\sqrt{1-x^2}}\left(\begin{array}{l} -2x\left(\frac{15x^2-3}{4}\sqrt{1-x^2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^3-13x}{2\sqrt{1-x^2}}\right) \\ \quad +4\left(\frac{3}{2}x\sqrt{1-x^2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-2}{\sqrt{1-x^2}}\right) \end{array}\right)$$
$$= \left(-\frac{15x^3-3x}{2}+6x\right)\ln\left(\frac{1+x}{1-x}\right)-\frac{15x^4-13x^2}{1-x^2}+\frac{12x^2-8}{1-x^2}$$
$$= -\frac{15x^3-15x}{2}\ln\left(\frac{1+x}{1-x}\right)-\frac{15x^4-25x^2+8}{1-x^2}$$
$$= \frac{15x(1-x^2)}{2}\ln\left(\frac{1+x}{1-x}\right)-\frac{15x^4-25x^2+8}{1-x^2} \tag{2-23}$$

$$
\begin{aligned}
Q_{33}(x) &= \frac{-(3-2)xQ_{32}(x)+(3+2)Q_{22}(x)}{\sqrt{1-x^2}} \\
&= \frac{1}{\sqrt{1-x^2}}\left(\begin{array}{l} -x\left(\dfrac{15x\left(1-x^2\right)}{2}\ln\left(\dfrac{1+x}{1-x}\right)-\dfrac{15x^4-25x^2+8}{1-x^2}\right) \\ \quad +5\left(\dfrac{3\left(1-x^2\right)}{2}\ln\left(\dfrac{1+x}{1-x}\right)-\dfrac{3x^3-5x}{1-x^2}\right) \end{array}\right) \\
&= \left(-\frac{15x^2}{2}+\frac{15}{2}\right)\sqrt{1-x^2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^5-25x^5+8x-15x^3+25x}{\left(1-x^2\right)^{3/2}} \\
&= \frac{15}{2}\left(1-x^2\right)^{3/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^5-40x^3+33x}{\left(1-x^2\right)^{3/2}}
\end{aligned}
\tag{2-24}
$$

となる．同様に式(2-18) を使って，$|x|>1,\quad n=1,2,3,\quad m=1,2,3$ の $Q_{nm}(x)$ を計算すると，

$$
\begin{aligned}
Q_{11}(x) &= \frac{(1-0)xQ_{10}(x)-(1+0)Q_{00}(x)}{\sqrt{x^2-1}} \\
&= \frac{x\left(xQ_0(x)-1\right)-Q_0(x)}{\sqrt{x^2-1}} = \frac{\left(x^2-1\right)Q_0(x)-x}{\sqrt{x^2-1}} \\
&= \sqrt{x^2-1}\,Q_0(x)-\frac{x}{\sqrt{x^2-1}}
\end{aligned}
\tag{2-25}
$$

$$
\begin{aligned}
Q_{21}(x) &= \frac{(2-0)xQ_{20}(x)-(2+0)Q_{10}(x)}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}\left(2x\left(\frac{3x^2-1}{2}Q_0(x)-\frac{3x}{2}\right)-2\left(xQ_0(x)-1\right)\right) \\
&= \frac{1}{\sqrt{x^2-1}}\left(\left(3x^3-x-2x\right)Q_0(x)-3x^2+2\right) \\
&= \frac{1}{\sqrt{x^2-1}}\left(3x\left(x^2-1\right)Q_0(x)-\left(3x^2-2\right)\right) \\
&= 3x\sqrt{x^2-1}\,Q_0(x)-\frac{3x^2-2}{\sqrt{x^2-1}}
\end{aligned}
\tag{2-26}
$$

$$
\begin{aligned}
Q_{22}(x) &= \frac{(2-1)xQ_{21}(x)-(2+1)Q_{11}(x)}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}\left(\begin{array}{l} x\left(\dfrac{3}{2}x\sqrt{1-x^2}\ln\left(\dfrac{x+1}{x-1}\right)-\dfrac{3x^2-2}{\sqrt{x^2-1}}\right) \\ \quad -3\left(\dfrac{1}{2}\sqrt{1-x^2}\ln\left(\dfrac{x+1}{x-1}\right)-\dfrac{x}{\sqrt{x^2-1}}\right)\end{array}\right) \\
&= \left(\frac{3}{2}x^2-\frac{3}{2}\right)\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^3-2x}{x^2-1}+\frac{3x}{x^2-1} \\
&= \frac{3\left(x^2-1\right)}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^3-2x-3x}{x^2-1} \\
&= \frac{3\left(x^2-1\right)}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^3-5x}{x^2-1}
\end{aligned} \tag{2-27}
$$

$$
\begin{aligned}
Q_{31}(x) &= \frac{(3-0)xQ_{30}(x)-(3+0)Q_{20}(x)}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}\left(3x\left(\frac{5x^3-3x}{2}Q_0(x)-\frac{15x^2-4}{6}\right)-3\left(\frac{3x^2-1}{2}Q_0(x)-\frac{3x}{2}\right)\right) \\
&= \frac{1}{\sqrt{x^2-1}}\left(\frac{3\left(5x^4-3x^2-3x^2+1\right)}{2}Q_0(x)-\frac{15x^3-4x-9x}{2}\right) \\
&= \frac{1}{\sqrt{x^2-1}}\left(\frac{3\left(5x^4-6x^2+1\right)}{2}Q_0(x)-\frac{15x^3-13x}{2}\right) \\
&= \frac{1}{\sqrt{x^2-1}}\left(\frac{3\left(x^2-1\right)\left(5x^2-1\right)}{2}Q_0(x)-\frac{15x^3-13x}{2}\right) \\
&= \frac{15x^2-3}{2}\sqrt{x^2-1}\,Q_0(x)-\frac{15x^3-3x}{\sqrt{x^2-1}}
\end{aligned} \tag{2-28}
$$

$$
\begin{aligned}
Q_{32}(x) &= \frac{(3-1)xQ_{31}(x)-(3+1)Q_{21}(x)}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}\left(\begin{array}{l} 2x\left(\dfrac{15x^2-3}{4}\sqrt{x^2-1}\ln\left(\dfrac{x+1}{x-1}\right)-\dfrac{15x^3-13x}{2\sqrt{x^2-1}}\right) \\ -4\left(\dfrac{3x}{2}\sqrt{x^2-1}\ln\left(\dfrac{x+1}{x-1}\right)-\dfrac{3x^2-2}{\sqrt{x^2-1}}\right)\end{array}\right) \\
&= \left(\frac{15x^3-3x}{2}-6x\right)\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^4-13x^2}{x^2-1}+\frac{12x^2-8}{x^2-1} \\
&= \frac{15x^3-15x}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^4-25x^2+8}{x^2-1} \\
&= \frac{15x\left(x^2-1\right)}{2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^4-25x^2+8}{x^2-1}
\end{aligned}
\tag{2-29}
$$

$$
\begin{aligned}
Q_{33}(x) &= \frac{(3-2)xQ_{32}(x)-(3+2)Q_{22}(x)}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}\left(\begin{array}{l} x\left(\dfrac{15x\left(x^2-1\right)}{2}\ln\left(\dfrac{x+1}{x-1}\right)-\dfrac{15x^4-25x^2+8}{x^2-1}\right) \\ -5\left(\dfrac{3\left(1-x^2\right)}{2}\ln\left(\dfrac{x+1}{x-1}\right)-\dfrac{3x^3-5x}{x^2-1}\right)\end{array}\right) \\
&= \left(\frac{15x^2}{2}-\frac{15}{2}\right)\sqrt{x^2-1}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^5-25x^5+8x-15x^3+25x}{\left(x^2-1\right)^{3/2}} \\
&= \frac{15}{2}\left(x^2-1\right)^{3/2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^5-40x^3+33x}{\left(x^2-1\right)^{3/2}}
\end{aligned}
\tag{2-30}
$$

となる．

3.3 第 2 種ルジャンドル陪関数のまとめ

$$Q_{11}(x)=\frac{1}{2}\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{x}{\left(1-x^2\right)^{1/2}}$$

$$Q_{21}(x)=\frac{3x}{2}\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-2}{\left(1-x^2\right)^{1/2}}$$

$$Q_{22}(x)=\frac{3}{2}\left(1-x^2\right)\ln\left(\frac{1+x}{1-x}\right)-\frac{3x^3-5x}{1-x^2}$$

$$Q_{31}(x)=\frac{15x^2-3}{4}\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^3-13x}{2\left(1-x^2\right)^{1/2}}$$

$$Q_{32}(x)=\frac{15x}{2}\left(1-x^2\right)\ln\left(\frac{1+x}{1-x}\right)-\frac{15x^4-25x^2+8}{1-x^2}$$

$$Q_{33}(x)=\frac{15}{2}\left(1-x^2\right)^{3/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^5-40x^3+33x}{\left(1-x^2\right)^{3/2}}$$

表 3　第 2 種ルジャンドル陪関数 $Q_{nm}(x)$　$|x|<1$

$$Q_{11}(x)=\frac{1}{2}\left(x^2-1\right)^{1/2}\ln\left(\frac{x+1}{x-1}\right)-\frac{x}{\left(x^2-1\right)^{1/2}}$$

$$Q_{21}(x)=\frac{3x}{2}\left(x^2-1\right)^{1/2}\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^2-2}{\left(x^2-1\right)^{1/2}}$$

$$Q_{22}(x)=\frac{3}{2}\left(x^2-1\right)\ln\left(\frac{x+1}{x-1}\right)-\frac{3x^3-5x}{x^2-1}$$

$$Q_{31}(x)=\frac{15x^2-3}{4}\left(x^2-1\right)^{1/2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^3-13x}{2\left(x^2-1\right)^{1/2}}$$

$$Q_{32}(x)=\frac{15x}{2}\left(x^2-1\right)\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^4-25x^2+8}{x^2-1}$$

$$Q_{33}(x)=\frac{15}{2}\left(x^2-1\right)^{3/2}\ln\left(\frac{x+1}{x-1}\right)-\frac{15x^5-40x^3+33x}{\left(x^2-1\right)^{3/2}}$$

表 4　第 2 種ルジャンドル陪関数 $Q_{nm}(x)$　$|x|>1$

$|x|<1$ における $Q_{nm}(x)$ を下図に示す.

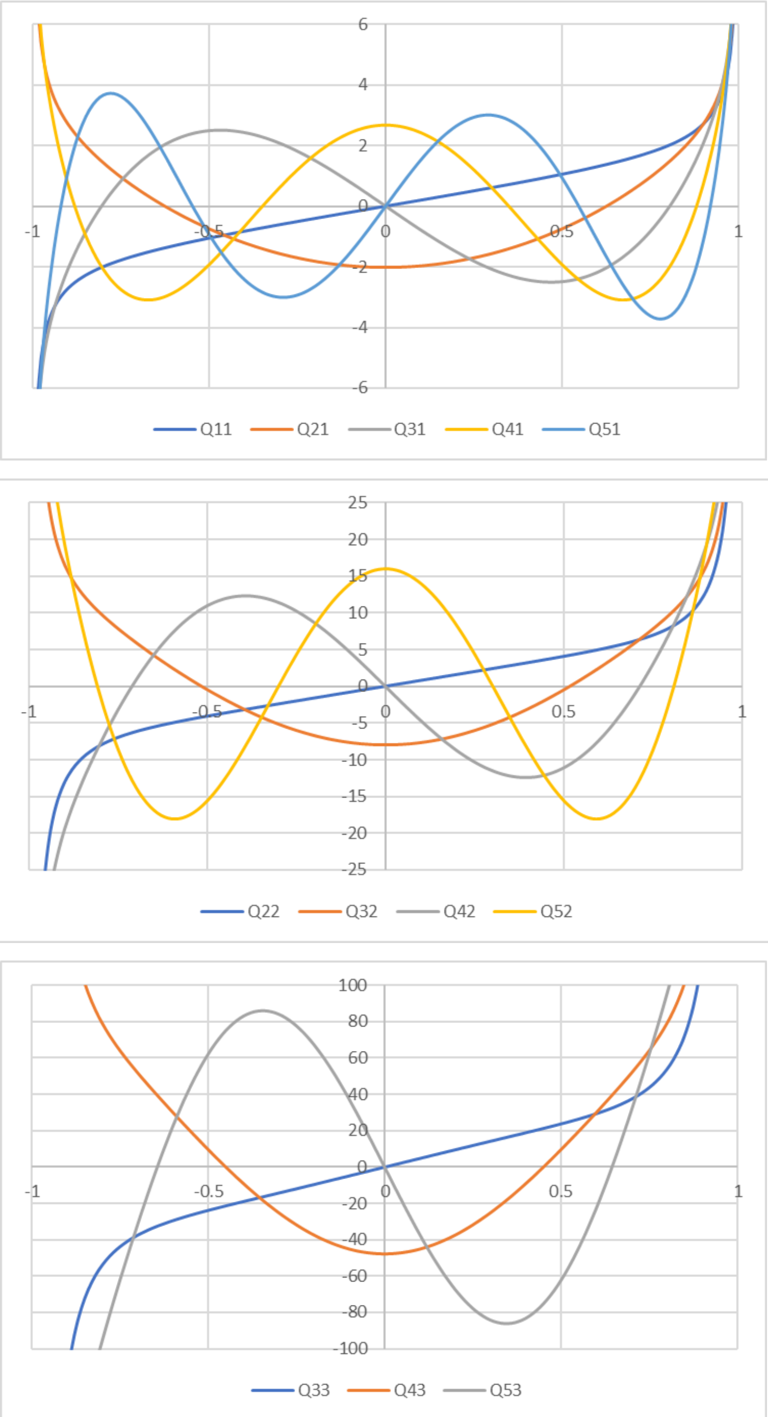


図 4　第 2 種ルジャンドル陪関数 $Q_{nm}(x)$　$|x|<1$

$Q_{nm}(x)$ は $x\to\pm1$ では正または負の無限大となる．すなわち，$x=\pm1$ は特異点である．$n+m$ が偶数の場合は奇関数，奇数の場合は偶関数である．$|x|\le2$ における $Q_{nm}(x)$ を下図に示す．

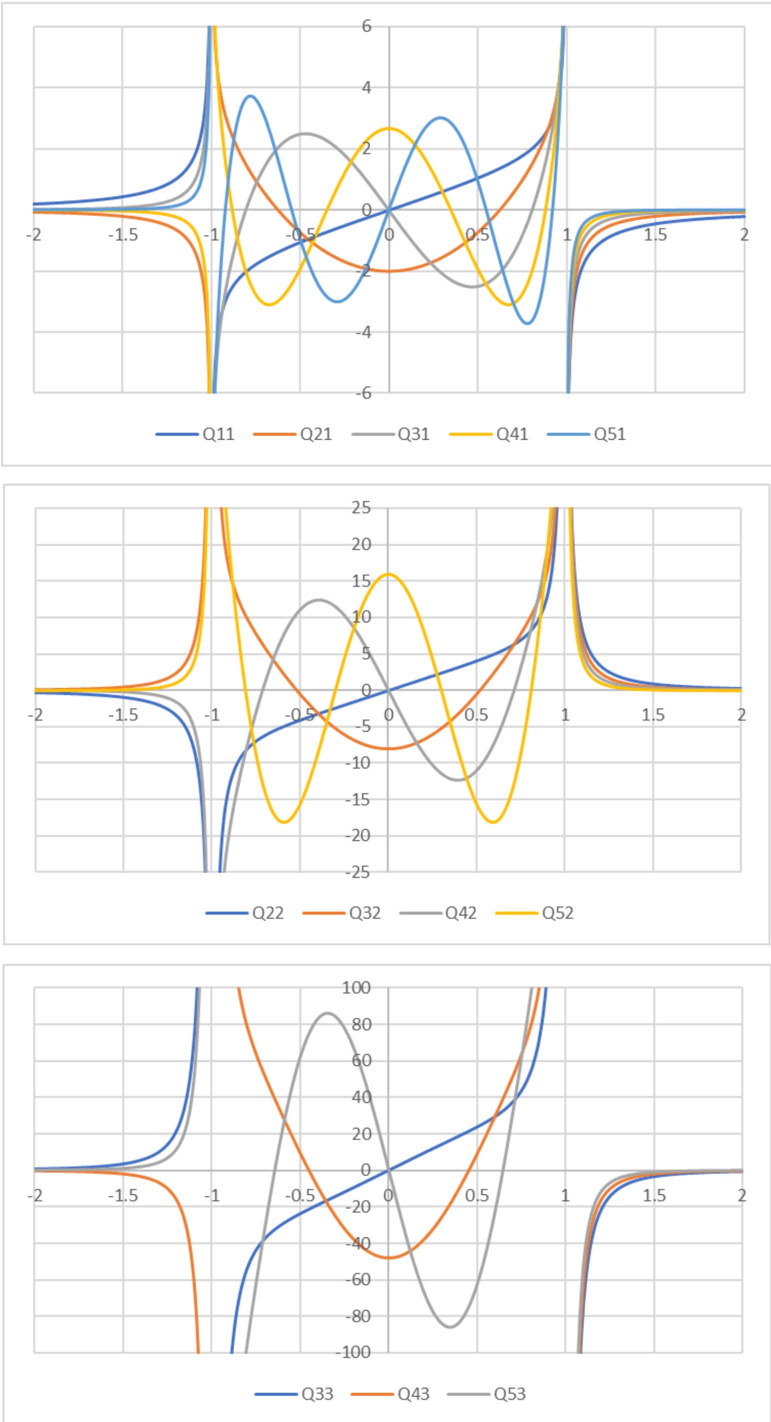


図 5　第 2 種ルジャンドル陪関数 $Q_{nm}(x)$　$|x| \le 2$

$1 \le x \le 10$　の $Q_{n,1}(x)$　の絶対値を対数目盛でプロットすると下図のようになる． $Q_{nm}(x)$　の絶対値は単調減少して， $x \to \pm\infty$　では 0 となる．

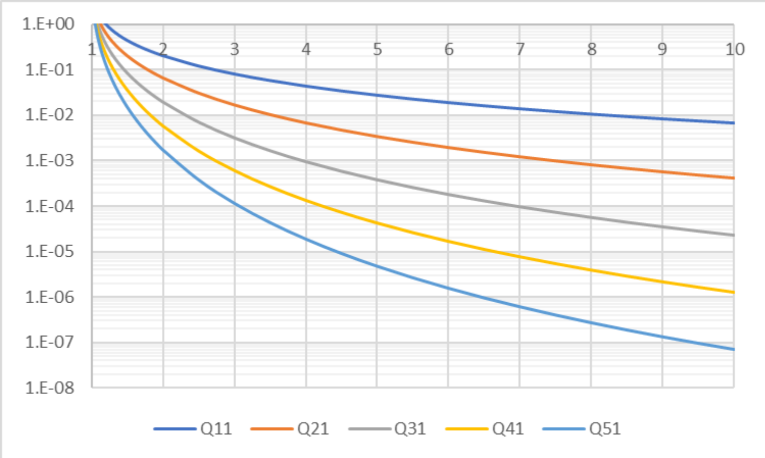


図 6　第 2 種ルジャンドル陪関数 $Q_{nm}(x)$　$1<x\le 10$

$x>>1$　の場合，$Q_{nm}(x)$ は次式のように表され，$-(n+1)$ 次式に近づき，0 に漸近する．

$$Q_{nm}(x)=(-1)^m\frac{(n+m)!}{(2n+1)!!}\cdot\frac{1}{x^{n+1}}+O\left(x^{-n-3}\right) \tag{2-31}$$

ここに，$(2n-1)!!=(2n-1)(2n-3)(2n-5)\cdots 3\cdot 1$　である．$O\left(x^{-n-3}\right)$　は最高次が $(-n-3)$ 次のなんらかの関数であり，第 1 項と比較すると 2 次小さい．具体的には，

$$\begin{aligned}
Q_{11}(x)&=-\frac{2}{3x^2}+O\left(x^{-4}\right)\\
Q_{21}(x)&=-\frac{2}{5x^3}+O\left(x^{-5}\right)\\
Q_{22}(x)&=\frac{8}{5x^3}+O\left(x^{-5}\right)\\
Q_{31}(x)&=-\frac{8}{35x^4}+O\left(x^{-6}\right)\\
Q_{32}(x)&=\frac{8}{7x^4}+O\left(x^{-6}\right)\\
Q_{33}(x)&=-\frac{48}{7x^4}+O\left(x^{-6}\right)
\end{aligned} \tag{2-32}$$

である．

3.4 第 2 種ルジャンドル陪関数の証明

$|x|<1,\quad n=1,2,3,\quad m=1,2,3$　の $Q_{nm}(x)$　をルジャンドル陪微分方程式(1-2) に実際に代入する．微分計算の途中は省略する．

$Q_{11}(x)=\frac{1}{2}\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{x}{\left(1-x^2\right)^{1/2}}$　の証明

$$\frac{d}{dx}Q_{11}(x)=\frac{-x}{2\left(1-x^2\right)^{1/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{2-x^2}{\left(1-x^2\right)^{3/2}}$$

$$\frac{d^2}{dx^2}Q_{11}(x)=\frac{-1}{2\left(1-x^2\right)^{3/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x}{\left(1-x^2\right)^{5/2}}$$

$$\begin{aligned}
&\left(1-x^2\right)\frac{d^2Q_{11}(x)}{dx^2}-2x\frac{dQ_{11}(x)}{dx}+\left(1\cdot 2-\frac{1^2}{1-x^2}\right)Q_{11}(x)\\
&=\left(1-x^2\right)\left(\frac{-1}{2\left(1-x^2\right)^{3/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x}{\left(1-x^2\right)^{5/2}}\right)\\
&\quad-2x\cdot\left(\frac{-x}{2\left(1-x^2\right)^{1/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{2-x^2}{\left(1-x^2\right)^{3/2}}\right)\\
&\quad+\left(2-\frac{1}{1-x^2}\right)\cdot\left(1-x^2\right)^{1/2}\left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{x}{1-x^2}\right)\\
&=\left(\frac{-1}{2\left(1-x^2\right)^{1/2}}+\frac{x^2}{\left(1-x^2\right)^{1/2}}+\frac{1-2x^2}{2\left(1-x^2\right)^{1/2}}\right)\ln\left(\frac{1+x}{1-x}\right)\\
&\quad+\frac{3x}{\left(1-x^2\right)^{3/2}}+\frac{-4x+2x^3}{\left(1-x^2\right)^{3/2}}+\frac{x\left(1-2x^2\right)}{\left(1-x^2\right)^{3/2}}=0
\end{aligned}\tag{B1}$$

$Q_{21}(x)=\frac{3}{2}x\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-2}{\left(1-x^2\right)^{1/2}}$　の証明

$$\frac{d}{dx}Q_{21}(x)=\frac{3-6x^2}{2\left(1-x^2\right)^{1/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{-6x^3+7x}{\left(1-x^2\right)^{3/2}}$$

$$\frac{d^2}{dx^2}Q_{21}(x)=\frac{6x^3-9x}{2\left(1-x^2\right)^{3/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{6x^4-13x^2+10}{\left(1-x^2\right)^{5/2}}$$

$$
\begin{aligned}
&\left(1-x^2\right)\frac{d^2Q_{21}(x)}{dx^2}-2x\frac{dQ_{21}(x)}{dx}+\left(2\cdot3-\frac{1^2}{1-x^2}\right)Q_{21}(x)\\
&=\left(1-x^2\right)\left(\frac{6x^3-9x}{2\left(1-x^2\right)^{3/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{6x^4-13x^2+10}{\left(1-x^2\right)^{5/2}}\right)\\
&\quad-2x\cdot\left(\frac{3-6x^2}{2\left(1-x^2\right)^{1/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{-6x^3+7x}{\left(1-x^2\right)^{3/2}}\right)\\
&\quad+\left(6-\frac{1}{1-x^2}\right)\cdot\left(\frac{3}{2}x\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{3x^2-2}{\left(1-x^2\right)^{1/2}}\right)\\
&=\left(\frac{6x^3-9x}{2\left(1-x^2\right)^{1/2}}+\frac{-3x+6x^3}{\left(1-x^2\right)^{1/2}}+\frac{3}{2}x\frac{5-6x^2}{\left(1-x^2\right)^{1/2}}\right)\ln\left(\frac{1+x}{1-x}\right)\\
&\quad+\frac{6x^4-13x^2+10}{\left(1-x^2\right)^{3/2}}+\frac{12x^4-14x^2}{\left(1-x^2\right)^{3/2}}+\frac{\left(5-6x^2\right)\left(3x^2-2\right)}{\left(1-x^2\right)^{3/2}}\\
&=\frac{6x^3-9x-6x+12x^3+15x-18x^3}{2\left(1-x^2\right)^{1/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{18x^4-27x^2+10-18x^4+27x^2-10}{\left(1-x^2\right)^{3/2}}\\
&=0
\end{aligned}
\tag{B2}
$$

$Q_{22}(x)=\frac{3}{2}\left(1-x^2\right)\ln\left(\frac{1+x}{1-x}\right)-\frac{3x^3-5x}{1-x^2}$ の証明

$$
\frac{d}{dx}Q_{22}(x)=-3x\ln\left(\frac{1+x}{1-x}\right)+\frac{6x^4-10x^2+8}{\left(1-x^2\right)^2}
$$

$$
\frac{d^2}{dx^2}Q_{22}(x)=-3\ln\left(\frac{1+x}{1-x}\right)+\frac{-6x^5+16x^3+6x}{\left(1-x^2\right)^3}
$$

$$
\begin{aligned}
&\left(1-x^2\right)\frac{d^2Q_{22}(x)}{dx^2}-2x\frac{dQ_{22}(x)}{dx}+\left(2\cdot 3-\frac{2^2}{1-x^2}\right)Q_{22}(x)\\
&=\left(1-x^2\right)\left(-3\ln\left(\frac{1+x}{1-x}\right)+\frac{-6x^5+16x^3+6x}{\left(1-x^2\right)^3}\right)\\
&\quad-2x\cdot\left(-3x\ln\left(\frac{1+x}{1-x}\right)+\frac{6x^4-10x^2+8}{\left(1-x^2\right)^2}\right)\\
&\quad+\left(6-\frac{4}{1-x^2}\right)\left(\frac{3}{2}\left(1-x^2\right)\ln\left(\frac{1+x}{1-x}\right)+\frac{-3x^3+5x}{1-x^2}\right)\\
&=\left(-3+3x^2+6x^2+9\left(1-x^2\right)-6\right)\ln\left(\frac{1+x}{1-x}\right)\\
&\quad+\frac{-6x^5+16x^3+6x}{\left(1-x^2\right)^2}+\frac{-12x^5+20x^3-16x}{\left(1-x^2\right)^2}+\frac{2-6x^2}{1-x^2}\cdot\frac{-3x^3+5x}{1-x^2}\\
&=\frac{-18x^5+36x^3-10x-6x^3+10x+18x^5-30x^2}{\left(1-x^2\right)^2}=0
\end{aligned}
\tag{B3}
$$

$Q_{31}(x)=\dfrac{15x^2-3}{4}\left(1-x^2\right)^{1/2}\ln\left(\dfrac{1+x}{1-x}\right)+\dfrac{15x^3-13x}{2\left(1-x^2\right)^{1/2}}$ の証明

$$
\frac{d}{dx}Q_{31}(x)=\frac{-45x^3+33x}{4\left(1-x^2\right)^{1/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{-45x^4+63x^2-16}{2\left(1-x^2\right)^{3/2}}
$$

$$
\frac{d^2}{dx^2}Q_{31}(x)=\frac{90x^4-135x^2+33}{4\left(1-x^2\right)^{3/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{90x^5-195x^3+111x}{2\left(1-x^2\right)^{5/2}}
$$

$$
\begin{aligned}
&\left(1-x^2\right)\frac{d^2Q_{31}(x)}{dx^2}-2x\frac{dQ_{31}(x)}{dx}+\left(3\cdot 4-\frac{1^2}{1-x^2}\right)Q_{31}(x)\\
&=\left(1-x^2\right)\left(\frac{90x^4-135x^2+33}{4\left(1-x^2\right)^{3/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{90x^5-195x^3+111x}{2\left(1-x^2\right)^{5/2}}\right)\\
&\quad -2x\cdot\left(\frac{-45x^3+33x}{4\left(1-x^2\right)^{1/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{-45x^4+63x^2-16}{2\left(1-x^2\right)^{3/2}}\right)\\
&\quad +\left(12-\frac{1}{1-x^2}\right)\left(\frac{15x^2-3}{4}\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^3-13x}{2\left(1-x^2\right)^{1/2}}\right)\\
&=\left(\begin{array}{l}\dfrac{90x^4-135x^2+33}{4\left(1-x^2\right)^{1/2}}-\dfrac{-90x^4+66x^2}{4\left(1-x^2\right)^{1/2}}\\ \quad+\dfrac{3\left(15x^2-3\right)\left(1-x^2\right)}{\left(1-x^2\right)^{1/2}}-\dfrac{15x^2-3}{4\left(1-x^2\right)^{1/2}}\end{array}\right)\ln\left(\frac{1+x}{1-x}\right)\\
&\quad +\frac{90x^5-195x^3+111x}{2\left(1-x^2\right)^{3/2}}+\frac{45x^5-63x^3+16x}{\left(1-x^2\right)^{3/2}}+\frac{11-12x^2}{1-x^2}\cdot\frac{15x^3-13x}{2\left(1-x^2\right)^{1/2}}\\
&=\frac{180x^4-216x^2+36+\left(180x^2-36\right)\left(1-x^2\right)}{4\left(1-x^2\right)^{1/2}}\ln\left(\frac{1+x}{1-x}\right)\\
&\quad +\frac{90x^5-195x^3+111x+90x^5-126x^3+32x-180x^5+321x^3-143x}{2\left(1-x^2\right)^{3/2}}=0
\end{aligned}
\tag{B4}
$$

$Q_{32}(x)=\dfrac{15}{2}x\left(1-x^2\right)\ln\left(\dfrac{1+x}{1-x}\right)-\dfrac{15x^4-25x^2+8}{1-x^2}$ の証明

$$
\frac{d}{dx}Q_{32}(x)=\frac{15-45x^2}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{45x^5-90x^3+49x}{\left(1-x^2\right)^2}
$$

$$
\frac{d^2}{dx^2}Q_{32}(x)=-45x\ln\left(\frac{1+x}{1-x}\right)+\frac{-90x^6+240x^4-198x^2+64}{\left(1-x^2\right)^3}
$$

$$\left(1-x^2\right)\frac{d^2Q_{32}(x)}{dx^2}-2x\frac{dQ_{32}(x)}{dx}+\left(3\cdot 4-\frac{2^2}{1-x^2}\right)Q_{32}(x)$$

$$=\left(1-x^2\right)\left(-45x\ln\left(\frac{1+x}{1-x}\right)+\frac{-90x^6+240x^4-198x^2+64}{\left(1-x^2\right)^3}\right)$$

$$-2x\cdot\left(\frac{15-45x^2}{2}\ln\left(\frac{1+x}{1-x}\right)+\frac{45x^5-90x^3+49x}{\left(1-x^2\right)^2}\right)$$

$$+\left(12-\frac{4}{1-x^2}\right)\left(\frac{15}{2}x\left(1-x^2\right)\ln\left(\frac{1+x}{1-x}\right)+\frac{-15x^4+25x^2-8}{1-x^2}\right)$$

$$=\left(-45x+45x^3-15x+45x^3+90x-90x^3-30x\right)\ln\left(\frac{1+x}{1-x}\right)$$

$$+\frac{-90x^6+240x^4-198x^2+64}{\left(1-x^2\right)^2}+\frac{-90x^6+180x^4-98x^2}{\left(1-x^2\right)^2}$$

$$+\frac{8-12x^2}{1-x^2}\cdot\frac{-15x^4+25x^2-8}{1-x^2}$$

$$=\frac{-180x^6+420x^4-296x^2+64-120x^4+200x^2-64+180x^6-300x^4+96x^2}{\left(1-x^2\right)^2}$$

$$=0 \qquad \text{(B5)}$$

$Q_{33}(x)=\frac{15}{2}\left(1-x^2\right)^{3/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^5-40x^3+33x}{\left(1-x^2\right)^{3/2}}$ の証明

$$\frac{d}{dx}Q_{33}(x)=-\frac{45}{2}x\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{-45x^6+120x^4-99x^2+48}{\left(1-x^2\right)^{5/2}}$$

$$\frac{d^2}{dx^2}Q_{33}(x)=-\frac{45-90x^2}{2\left(1-x^2\right)^{1/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{90x^7-285x^5+318x^3-3x}{\left(1-x^2\right)^{7/2}}$$

$$
\begin{aligned}
&\left(1-x^2\right)\frac{d^2Q_{33}(x)}{dx^2}-2x\frac{dQ_{33}(x)}{dx}+\left(3\cdot 4-\frac{3^2}{1-x^2}\right)Q_{33}(x)\\
&=\left(1-x^2\right)\left(-\frac{45-90x^2}{2\left(1-x^2\right)^{1/2}}\ln\left(\frac{1+x}{1-x}\right)+\frac{90x^7-285x^5+318x^3-3x}{\left(1-x^2\right)^{7/2}}\right)\\
&\quad-2x\cdot\left(-\frac{45}{2}x\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{-45x^6+120x^4-99x^2+48}{\left(1-x^2\right)^{5/2}}\right)\\
&\quad+\left(12-\frac{9}{1-x^2}\right)\left(\frac{15}{2}\left(1-x^2\right)^{3/2}\ln\left(\frac{1+x}{1-x}\right)+\frac{15x^5-40x^3+33x}{\left(1-x^2\right)^{3/2}}\right)\\
&=\left(-\frac{45-90x^2}{2}\left(1-x^2\right)^{1/2}+45x^2\left(1-x^2\right)^{1/2}+\left(3-12x^2\right)\cdot\frac{15}{2}\left(1-x^2\right)^{1/2}\right)\ln\left(\frac{1+x}{1-x}\right)\\
&\quad+\frac{90x^7-285x^5+318x^3-3x}{\left(1-x^2\right)^{5/2}}+\frac{90x^7-240x^5+198x^3-96x}{\left(1-x^2\right)^{5/2}}\\
&\quad+\frac{\left(3-12x^2\right)\left(15x^5-40x^3+33x\right)}{\left(1-x^2\right)^{5/2}}\\
&=\left(-\frac{45-90x^2}{2}+45x^2+\left(3-12x^2\right)\cdot\frac{15}{2}\right)\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)\\
&\quad+\frac{180x^7-525x^5+516x^3-99x+45x^5-120x^3+99x-180x^7+480x^5-396x^3}{\left(1-x^2\right)^{5/2}}\\
&=\frac{-45+90x^2+90x^2+45-180x^2}{2}\left(1-x^2\right)^{1/2}\ln\left(\frac{1+x}{1-x}\right)=0
\end{aligned}
\tag{B6}
$$

確かにルジャンドル陪微分方程式を満足している．$|x|>1$ の場合も同様に証明できる．

参考文献

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